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The Two Techniques for Generating Perfect Fluid Sphere

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ABSTRACT

The Einstein field equation is the second order partial differential equation. It relates spacetime curvature to matter. In general, the exact solutions cannot be obtained because the equation is so complicated. One of the popular assumptions to reduce the complexity of the equation is that matter is a static, spherically symmetric, perfect fluid. In this way, the Einstein field equation is transformed to a second order differential equation with variable coefficients. In this paper, we are interested in solutions about regular singular points. Therefore, the method of Frobenius can be applied. Moreover, the reduced Einstein equation is of the Riccati form. With the property of the Riccati equation, we can find the general solutions if the particular solutions are specified.

Keywords: Differential equation with variable coefficients, Einstein field equation, method of Frobenius, perfect fluid, Riccati equation.

1. Introduction

The Einstein field equations describe how spacetime curves by the presence of matter and energy. They are the central equations in the general theory of relativity. Mathematically, the Einstein field equations are the second order partial differential equations. In fact, they are a set of 16 equations in our $(3+1)$-dimensional real world. Without any assumption or symmetry, the Einstein field equations cannot exactly be solved. To look for an exact solution, some assumptions have to be imposed to reduce the complexity of the Einstein field equations. An example of such assumptions is to model matter as a static and spherically symmetric perfect fluid. This assumption leads to the first two exact solutions to the Einstein field equations, which are known as the (exterior) Schwarzschild solution and the interior Schwarzschild solution Schwarzschild (1916). After this discovery, a static perfect fluid sphere became more popular, after which many exact solutions were found Bondi (1947), Boonserm (2006), Boonserm and Visser (2008), Boonserm and Weinfurtner (2005), Buchdahl (1959), Delgaty and Lake (1998), Herrera (2008), Kramer (1980), Lake (2003), Martin and Visser (2004), Rahman and Visser (2002).

Exact solutions can be obtained in many different ways. Some tried to solve the Einstein field equations directly with the aid of some assumptions. Some used special techniques to obtain exact solutions without solving the Einstein field equations Boonserm and Visser (2016), Boonserm and Thairatana, Jongjittanon and Ngampitipan (2016), Thairatana (2013). Some generated new solutions from previously known solutions using the property of the Riccati equation Boonserm and Weinfurtner (2007), Jongjittanon (2015), Kauser and Islam, Kinreewong (2015). Some applied the Frobenius method Piaggio (2008), Riley and Bence (2006) to solve the Klein-Gordon equation on a curved background given by the Einstein field equation Konoplya and Zhidenko (2011) and the generalized Einstein field equation in higher derivative gravity theories Perkins (2016). In this paper, we will use the Frobenius method and the properties of the Riccati equation to obtain exact solutions.

This paper is organized as follows. The assumption of perfect fluid spheres is imposed in section 2. The solutions to the Einstein field equations obtained by the Frobenius method and the properties of the Riccati equation are given in sections 3 and 4, respectively. A comparison between the two methods is made in section 5. Finally, a concluding remark is provided in section 6.

2. Perfect fluid spheres

The curvature of spacetime is described by the Einstein equations

$$G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu}, \tag{1}$$

where G^{μ}_{ν} is the Einstein tensor and T^{μ}_{ν} is the energyâ-momentum tensor, which takes the form

$$T^{\mu}_{\nu} = \text{diag}(-\rho, p_r, p_t, p_t). \tag{2}$$

We are interested in perfect fluid as the source of the energyâ-momentum tensor. One of the properties of being a perfect fluid is its pressures in all directions are the same. That is $p_r = p_t$ or, in terms of the components of the energyâ-momentum tensor,

$$T^{1}_{1} = T^{2}_{2}. \tag{3}$$

From (1), the above condition leads to

$$G^{1}_{1} = G^{2}_{2}. \tag{4}$$

The metric of the spacetime is given by

$$ds^2 = \zeta^2(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5}$$

Applying this metric to (4), we obtain

$$2r^2B(r)\zeta''(r) + [r^2B'(r) - 2rB(r)]\zeta'(r) + [rB'(r) - 2B(r) + 2]\zeta(r) = 0. \tag{6}$$

This is a second order linear ordinary differential equation. Moreover, this equation must be satisfied when the condition of perfect fluid spheres is imposed.

3. Frobenius Method

To apply the Frobenius method, equation (6) is rewritten as

$$r^2\zeta''(r) + \frac{rB'(r) - 2B(r)}{2B(r)}r\zeta'(r) + \frac{rB'(r) - 2B(r) + 2}{2B(r)}\zeta(r) = 0. \tag{7}$$

Dividing the above equation by r^2 gives

$$\zeta''(r) + \frac{rB'(r) - 2B(r)}{2rB(r)}\zeta'(r) + \frac{rB'(r) - 2B(r) + 2}{2r^2B(r)}\zeta(r) = 0. \tag{8}$$

We see that the coefficients of $\zeta'(r)$ and $\zeta(r)$ are not analytic at $r = 0$. Thus, the Frobenius method can be applied to find a power series of the form

$$\zeta(r) = \sum_{k=0}^{\infty} A_k r^{k+n}, \tag{9}$$

where $A_0 \neq 0$. Differentiating the above solution gives

$$\zeta'(r) = \sum_{k=0}^{\infty} (k+n) A_k r^{k+n-1} \tag{10}$$

and

$$\zeta''(r) = \sum_{k=0}^{\infty} (k+n-1)(k+n) A_k r^{k+n-2}. \tag{11}$$

Moreover, expanding the coefficients of $r\zeta'(r)$ and $\zeta(r)$ in (7) in a power series gives

$$\frac{rB'(r) - 2B(r)}{2B(r)} = -1 + \sum_{i=1}^{\infty} p_i r^i \tag{12}$$

and

$$\frac{rB'(r) - 2B(r) + 2}{2B(r)} = \frac{1 - B(0)}{B(0)} + \sum_{i=1}^{\infty} q_i r^i. \tag{13}$$

Substituting (9) - (13) into (7), we obtain

$$\begin{aligned} &\left[n(n-1) - n + \frac{1 - B(0)}{B(0)} \right] A_0 r^n + \\ &\sum_{k=1}^{\infty} \left[(k+n-1)(k+n) - (k+n) + \frac{1 - B(0)}{B(0)} \right] A_k r^{k+n} + \\ &\left(\sum_{i=1}^{\infty} p_i r^i \right) \left[\sum_{k=0}^{\infty} (k+n) A_k r^{k+n} \right] + \left(\sum_{i=1}^{\infty} q_i r^i \right) \left(\sum_{k=0}^{\infty} A_k r^{k+n} \right) = 0. \end{aligned} \tag{14}$$

The indicial equation is given by

$$n(n-1) - n + \frac{1 - B(0)}{B(0)} = 0. \tag{15}$$

The roots of the above indicial equation is given by

$$n = \frac{2 \pm \sqrt{4 - 4[1 - B(0)]/B(0)}}{2} = 1 \pm \sqrt{1 - \frac{1 - B(0)}{B(0)}} = 1 \pm \sqrt{\frac{2B(0) - 1}{B(0)}}. \tag{16}$$

We choose $B(r) = -1/2 - br^2$, where b is constant. Substituting in equations (12) and (13), we obtain

$$\frac{rB'(r) - 2B(r)}{2B(r)} = -\frac{1}{1 + 2br^2} = -1 - \sum_{i=1}^{\infty} (-2b)^i r^{2i} \tag{17}$$

and

$$\frac{rB'(r) - 2B(r) + 2}{2B(r)} = -\frac{3}{1 + 2br^2} = -3 - 3\sum_{i=1}^{\infty} (-2b)^i r^{2i}. \tag{18}$$

From the chosen $B(r)$, we obtain $B(0) = -1/2$. Equation (16) becomes

$$n = -1, 3. \tag{19}$$

The two roots are separated by an integer. Consider the smaller root $n = -1$. Substituting in (14) gives

$$\sum_{k=1}^{\infty} [(k-2)(k-1) - (k-1) - 3] A_k r^{k-1} - \left[\sum_{i=1}^{\infty} (-2b)^i r^{2i}\right] \left[\sum_{k=0}^{\infty} (k-1) A_k r^{k-1}\right] - \left[3 \sum_{i=1}^{\infty} (-2b)^i r^{2i}\right] \left(\sum_{k=0}^{\infty} A_k r^{k-1}\right) = 0.$$

Rearranging the above equation, we obtain

$$\sum_{k=1}^{\infty} k(k-4) A_k r^{k-1} - \left[\sum_{i=1}^{\infty} (-2b)^i r^{2i}\right] \left[\sum_{k=0}^{\infty} (k+2) A_k r^{k-1}\right] = 0. \tag{20}$$

Rearranging the above equation, we obtain

$$\begin{aligned} &-3A_1 + (-4A_2 + 4bA_0) r + (-3A_3 + 6bA_1) r^2 + \\ &\left(-8b^2 A_0 + 8bA_2\right) r^3 + \left(5A_5 - 12b^2 A_1 + 10bA_3\right) r^4 + \\ &\left(12A_6 + 16b^3 A_0 - 16b^2 A_2 + 12bA_4\right) r^5 + \ldots = 0. \end{aligned} \tag{21}$$

The coefficients of any power of r must be zero. We find that

$$A_1 = A_3 = A_5 = \ldots = 0 \tag{22}$$

and

$$\begin{aligned} A_2 &= bA_0 \\ A_6 &= -bA_4. \end{aligned} \tag{23}$$

Substituting in (9), we obtain

$$\begin{aligned} \zeta(r) &= \frac{A_0}{r} + bA_0 r + A_4 r^3 - bA_4 r^5 + \ldots \\ &= A_0\left(\frac{1}{r} + br\right) + A_4 r^3\left(1 - br^2 + \ldots\right), \end{aligned} \tag{24}$$

where A_0 and A_4 are arbitrary constants. This is the general solution to equation (7).

4. Riccati Equation

To transform equation (8) into the Riccati equation, we start by defining a new function

$$h(r) = \frac{\zeta'(r)}{\zeta(r)}. \tag{25}$$

Its first derivative is given by

$$h'(r) = \frac{\zeta''(r)}{\zeta(r)} - h^2(r). \tag{26}$$

Thus,

$$\frac{\zeta''(r)}{\zeta(r)} = h'(r) + h^2(r). \tag{27}$$

Substituting (25) and (27) into (8), we get

$$h'(r) + h^2(r) + \frac{rB'(r) - 2B(r)}{2rB(r)}h(r) + \frac{rB'(r) - 2B(r) + 2}{2r^2B(r)} = 0. \tag{28}$$

Rearranging the above equation gives

$$h'(r) = -\frac{rB'(r) - 2B(r) + 2}{2r^2B(r)} - \frac{rB'(r) - 2B(r)}{2rB(r)}h(r) - h^2(r). \tag{29}$$

From the chosen $B(r)$, the above equation becomes

$$h'(r) = \frac{3}{r^2 + 2br^4} + \frac{1}{r + 2br^3} h(r) - h^2(r). \tag{30}$$

This is the Riccati equation of which the general form is

$$A'(r) = q_0(r) + q_1(r)A(r) + q_2(r)A^2(r), \tag{31}$$

where $q_0(r) \neq 0$ and $q_2(r) \neq 0$. Comparing (31) with (30), we obtain

$$q_0(r) = \frac{3}{r^2 + 2br^4}, q_1(r) = \frac{1}{r + 2br^3}, q_2(r) = -1. \tag{32}$$

Let $h_0(r)$ satisfy (30). Then,

$$h'_0(r) = \frac{3}{r^2 + 2br^4} + \frac{1}{r + 2br^3} h_0(r) - h_0^2(r). \tag{33}$$

By the property of the Riccati equation, the other solution is given by

$$h(r) = h_0(r) + \frac{1}{z(r)}, \tag{34}$$

where $z(r)$ satisfies

$$z'(r) - \left[-\frac{1}{r + 2br^3} + 2h_0(r) \right] z(r) = 1. \tag{35}$$

The solution is given by

$$z(r) = \frac{1}{e^{\int P(r)dr}} \int e^{\int P(r)dr} dr, \tag{36}$$

where

$$P(r) = - \left[-\frac{1}{r + 2br^3} + 2h_0(r) \right]. \tag{37}$$

Thus,

$$z(r) = \sqrt{\frac{1 + 2br^2}{2br^2}} e^{\int 2h_0(r)dr} \int \sqrt{\frac{2br^2}{1 + 2br^2}} e^{-\int 2h_0(r)dr} dr. \tag{38}$$

From (34), we obtain

$$h(r) = h_0(r) + \sqrt{\frac{2br^2}{1 + 2br^2}} e^{-\int 2h_0(r)dr} \left[\int \sqrt{\frac{2br^2}{1 + 2br^2}} e^{-\int 2h_0(r)dr} dr \right]^{-1}. \tag{39}$$

From (25), $\zeta(r)$ is given by

$$\zeta(r) = ce^{\int h(r)dr}, \tag{40}$$

where c is an arbitrary constant. This is the other solution to equation (30). After knowing $h_0(r)$, we can know the explicit form of (40).

5. Comparing the Two Methods

From section 3, we can see that the general solution can be obtained using the Frobenius method. However, the method of Frobenius can be applied to a second order linear ordinary differential equation with some specific form.

On the other hand, the method of Riccati in section 4 can be applied to a first order nonlinear ordinary differential equation with any form. However, only knowing one solution, we can find the other solution. In fact, any second order linear ordinary differential equation can be transformed into the Riccati equation and vice versa. A single known solution to the Riccati equation can therefore be obtained through many methods for a second order linear ordinary differential equation including the Frobenius method.

6. Conclusion

The Einstein field equation is a second order nonlinear partial differential equation. When imposing the condition of perfect fluid spheres, the Einstein field equation can be transformed into a second order linear ordinary differential equation. After rearranging this equation, we can obtain the general solution by the method of Frobenius. Moreover, this second order linear ordinary differential equation can be transformed to the Riccati equation. After knowing one solution, we can obtain the other solution.

It is not obvious to conclude which of the two methods is more preferable. Sometimes, it is necessary to combine the two methods to obtain the general solution to a second order linear ordinary differential equation.

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